So the candidate points and the values of f at them are:

Candidate	Value of f	
$(0,0)$	1	
$\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$e^{-\frac{1}{4}}$	min
$\left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$	$e^{1/4}$	max
$\left(\frac{-1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$	$e^{1/4}$	max
$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$	$e^{-1/4}$	min

$e^{1/4} > 1$ since $e > 1$

$\Rightarrow e^{-1/4} < 1$

$\diamond$

Lecture 12

Two Constraints: Suppose we want to extremize $f(x,y,z)$ subject to two constraints: $g(x,y,z)=c$ & $h(x,y,z)=k$. Geometrically, we are extremizing f along the curve of intersection of $g=c$ & $h=k$. Now, we still have that ∇f is perpendicular to the curve of intersection at an extreme point, P, but it isn't necessarily perpendicular to both $g=c$ & $h=k$ at this point. However, since both ∇g & ∇h are perpendicular to the curve at this point, we find that $\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P)$. To summarize, we now have to solve

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = c \\ h(x,y,z) = k \end{cases}$$

Ex: Find the points on the conic section determined by $z^2 = x^2+y^2$ and $z = x+y+2$ which are closest to the origin.

Sol: Let $g = x^2+y^2-z^2$ & $h = x+y-z+2$.

The function we want to minimize subject to $g=0$ & $h=0$ is

$$f(x,y,z) = [d((0,0,0),(x,y,z))]^2 = x^2+y^2+z^2$$

(since distance is positive, a minimum of d^2 is a minimum of d.)

$\nabla f = \langle 2x, 2y, 2z\rangle$, $\nabla g = \langle 2x, 2y, -2z\rangle$, $\nabla h = \langle 1,1,-1\rangle$. So:

$$\begin{cases} 2x = \lambda 2x + \mu & (1) \rightarrow \mu = 2x(1-\lambda) \\ 2y = \lambda 2y + \mu & (2) \rightarrow \mu = 2y(1-\lambda) \\ 2z = -\lambda 2z - \mu & (3) \\ z^2 = x^2+y^2 & (4) \\ z = x+y+2 & (5) \end{cases}$$

$\rightarrow \boxed{x(1-\lambda) = y(1-\lambda)}$ (6)

(6) $\Rightarrow x = y$ or $\lambda = 1$

$\boxed{\lambda=1}$: Then $\mu = 0$, so, by (3), $2z = -2z \Rightarrow z = 0$

Thus, by (4): $0 = x^2+y^2 \Rightarrow x = y = 0$.

Plug this into (5): $0 = 0+0+2 = 2$. Contradiction! So $\lambda \neq 1$.

Thus $x = y$. Plugging this into (4) & (5), we get:

$$\begin{cases} z^2 = 2x^2 & (4') \\ z = 2x+2 & (5') \end{cases}$$

Plug (5') into (4'):

$z^2 = (2x+2)^2 = 4x^2+8x+4 = 2x^2$

$\Rightarrow 2x^2+8x+4 = 0 \Rightarrow x = \dfrac{-8 \pm \sqrt{64-32}}{4} = -2 \pm \sqrt{2} = y$

Use (57) to find z:

If $x = y = -2+\sqrt{2}$: $z = 2(-2+\sqrt{2})+2 = 2\sqrt{2}-2$

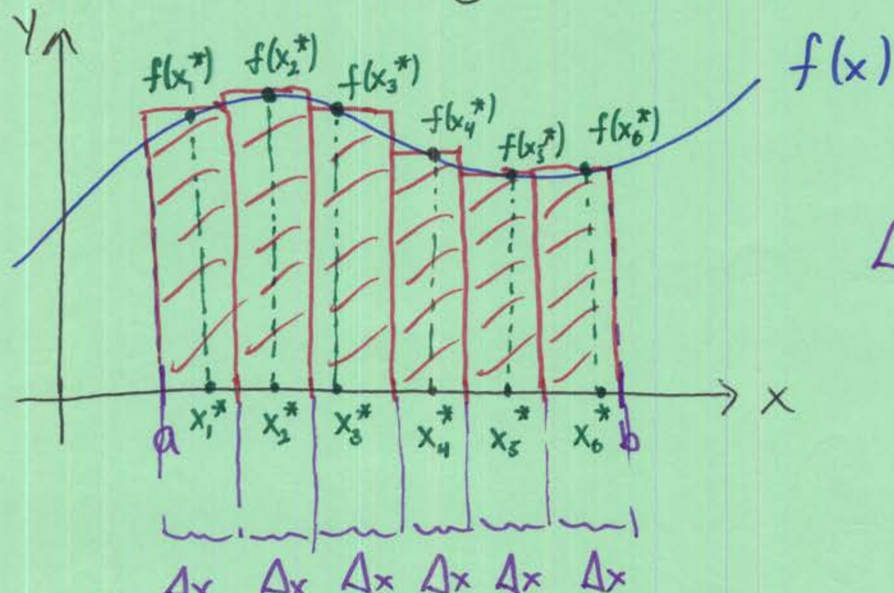
If $x = y = -2-\sqrt{2}$: $z = 2(-2-\sqrt{2})+2 = -2\sqrt{2}-2$

Candidate Points	Value
$(-2+\sqrt{2}, -2+\sqrt{2}, 2\sqrt{2}-2)$	$24-16\sqrt{2}$ ⟵ Closest point
$(-2-\sqrt{2}, -2-\sqrt{2}, -2\sqrt{2}-2)$	$24+16\sqrt{2}$

◇

15.1 - Double Integrals Over Rectangles

Let's recall how integrals are defined:



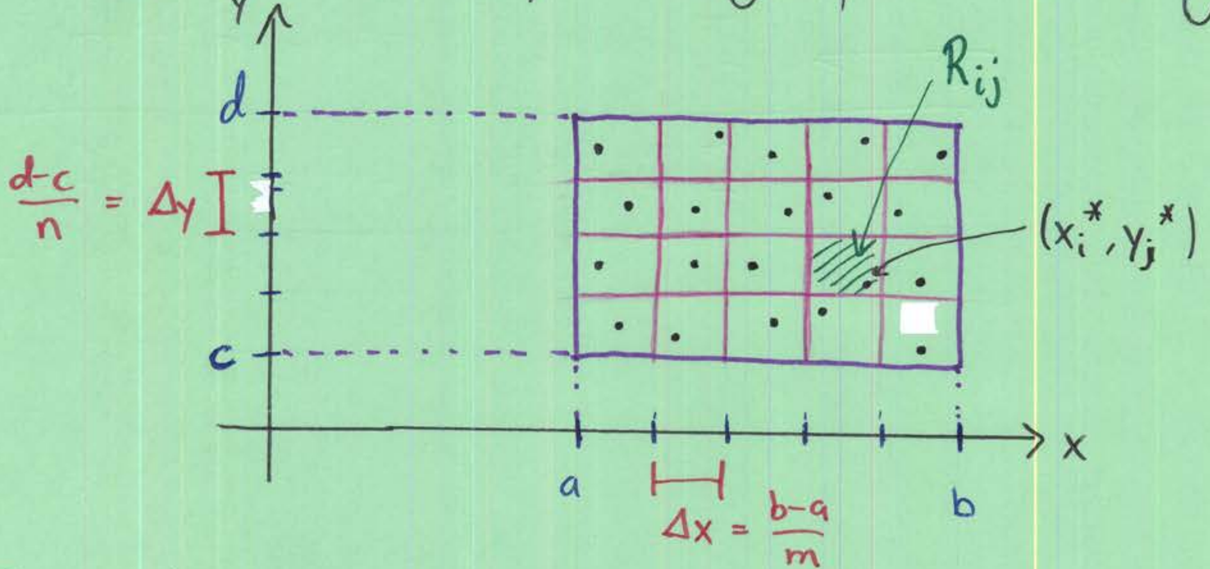
$$\Delta x = \frac{b-a}{6}$$

$$\text{Area} \approx \sum_{i=1}^{7} f(x_i^*)\Delta x$$

$$\text{Area} = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Double Integrals

Let's start with a simple region: a rectangle.
Let $R = [a,b] \times [c,d]$ and let $f = f(x,y)$ contain R in its domain. We'll also assume for now that $f \geq 0$ on R. We start by cutting up the rectangle:



In each subrectangle R_{ij} we choose a sample point (x_i^*, y_j^*), and over each R_{ij} construct a column of height $f(x_i^*, y_j^*)$. Adding up these volumes gives an approximation of the volume under f:

$$\mathrm{Vol} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \left(f(x_i^*, y_j^*) \Delta x \Delta y \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A$$

This is the <u>double Riemann sum</u>.
Now, of course, to get the actual volume we have to take finer and finer partitions ($\Delta x, \Delta y \to 0 \Leftrightarrow m, n \to \infty$)

So, $V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$. Now, none of this required $f \geqslant 0$ on R, so we get the final definition:

<u>Def</u>: The <u>double integral</u> of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$

if the limit exists.

<u>Facts</u>:

1) If $f(x,y) \geqslant 0$, then the volume V of the solid which lies above R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

2) $\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

3) $\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA$ (c a constant)

4) If $f(x,y) \geqslant g(x,y)$ for all (x,y) in R, then:

$$\iint_R f(x,y)\,dA \geqslant \iint_R g(x,y)\,dA.$$

15.2 - Iterated Integrals

This will give us a way of actually computing double integrals. Using the definition, let's take the "m limit" first. This amounts to integrating x first:

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty}\sum_{j=1}^{n}\left(\lim_{m\to\infty}\sum_{i=1}^{m} f(x_i^*, y_j^*)\Delta x\right)\Delta y$$

$$= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\int_a^b f(x, y_j^*)\,dx\right)\Delta y$$

$$= \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

doing the "n limit" first would give integrating y first: $\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx$

These are called <u>iterated integrals</u>. Notice that we are working from the <u>inside out</u> in these integrals. So, now we have the question of how to compute $\int_a^b f(x,y)\,dx$ or $\int_c^d f(x,y)\,dy$? The answer is <u>partial integration</u>, which are performed analogusly to partial derivatives.

12-7

Ex: Compute $\int_0^5 12x^2y^3\,dx$ and $\int_0^1 12x^2y^3\,dy$, and the associated indefinite integrals.

Sol:

First $\int 12x^2y^3\,dx = 4x^3y^3 + g(y)$ ← notice that this is a function of y since "x sees y as a constant"

$$\int_0^5 12x^2y^3\,dx = 4x^3y^3\Big|_0^5 = 4\cdot 5^3\cdot y^3 - 0 = 500y^3$$

$$\int 12x^2y^3\,dy = 3x^2y^4 + h(x)$$

$$\int_0^1 12x^2y^3\,dy = 3x^2y^4\Big|_0^1 = 3x^2\cdot 1^4 - 0 = 3x^2$$

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Lecture 13

13-1

Ex: Compute:

ⓐ $\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx$ ⓑ $\int_0^4\int_0^2 y^3e^{2x}\,dx\,dy$

Sol: ⓐ $\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx = \int_0^2 \frac{1}{4}y^4e^{2x}\Big|_0^4\,dx$

$= \int_0^2 64e^{2x}\,dx = 32e^{2x}\Big|_0^2 = 32(e^4-1)$

ⓑ $\int_0^4\int_0^2 y^3e^{2x}\,dx\,dy = \int_0^4 \frac{1}{2}y^3e^{2x}\Big|_0^2\,dy$

$= \int_0^4 \frac{1}{2}(e^4-1)y^3\,dy = \frac{1}{8}(e^4-1)y^4\Big|_0^4 = \frac{1}{8}(e^4-1)\cdot 256$

$= 32(e^4-1).$

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